Beads Hanging on A Ring

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Abstract

In this experiment, we study the physics of the motion of two beads on a ring. We determine the critical mass and angle at which the ring jumps upwards as a result of the effects of the beads on it. We use Newtonian mechanics, specifically Newton's 2nd law, in our analysis. In our experiment, we found the critical mass and angle to be related as follows, $\theta_c = 25^\circ \pm 3^\circ$ and $\frac{m}{M} = 1.52892 \pm 0.00004$, which corresponds with the results of our calculations.

Theory

Newton's law of motion helps us describe any kind of motion to a certain degree of certainty. We specifically use Newton's second law of motion which states the relationship between force and acceleration as follows, $F = ma$. In our experiment, the beads on the ring undergo a circular motion around a specified radius R. This motion is the result of the combined effect of the normal force due to the ring on the beads and the earth's gravity acting on the beads. The forces result in what's called centripetal acceleration, which keeps the beads on the circular trajectory.

For a ring of mass M on a thread with two bead of mass m sliding on it without friction simultaneously from the top of the ring, the ring rises if there is enough normal force acting from the beads upward and also enough to support the whole weight of the ring, which infers the tension from the thread at that point should be zero,

We first express the forces acting on each object relevant to our problem,

For each bead:

$$N + mg\cos\theta = \frac{mv^2}{R} \quad (1)$$

For the ring:

$$\hat{j} \quad T + 2N\cos\theta - Mg = 0, \quad \text{when } T = 0, \quad 2N\cos\theta = Mg \quad (2)$$

We then find the speed of the beads using conservation of energy,

$$\Delta K + \Delta U = 0 \implies \frac{1}{2}mv^2 + mgR(\cos\theta - 1) = 0$$

$$v^2 = 2gR(1 - \cos\theta) \quad (3)$$

We use eqn. (3) in eqn. (1) to solve for N,

$$N = mg(2 - 3\cos\theta) \quad (4)$$

We use the normal force from eqn. (4) to solve for the critical mass using eqn. (2),

We also label $\theta = \theta_c$ as we're looking at the critical condition for the ring to rise up,

To find the critical mass, we have to find θ_c at which we obtain the maximum value,

$$f(\theta_c) = (2 - 3\cos\theta_c)\cos\theta_c - \frac{M}{2m} = 0$$
$$\cos\theta_c = \frac{1}{3} + \sqrt{\frac{1}{9} - \frac{M}{6m}} \quad (5)$$

Using the above condition, we get the critical mass m to be the following at the maximum critical angle $\theta_c = \arccos(1/3)$,

$$m \geq \frac{3}{2}M \qquad (6)$$

Results and Analysis

Instruments and softwares used to conduct our measurements and build materials:

- Neiko 6" Digital Caliper: measuring the radius of the sphere and thickness of the ring
- Mettler Toledo XSR64 analytical balance: measure mass of the beads and ring
- TinkerCad and PRUSA: used to build and 3d print our ring and beads

For our experiment, we used the following parameters for our beads and rings:

- Mass of each bead m: $3.89820g \pm 0.00004g, \quad 3.89750g \pm 0.0004g$
- Mass of the ring M: $2.54940g \pm 0.00004g$
- Thickness of the ring d: $1.96mm \pm 0.02mm$
- Radius of each bead r: $10.07mm \pm 0.02mm, \quad 10.00mm \pm 0.02mm$
- Radius of the ring R: $210mm$

Note that for error propagation from our initial measurements, we use the following,

$$\Delta f \approx \frac{\delta f}{\delta x}\Delta x$$

Where Δf is the error propagated when f is evaluated at x with an error of Δx.

We calculate the ratio of the mass to check if it satisfies the constraint condition at which the ring starts to rise:

$$\frac{m_1}{M} = 1.52906 \pm 0.00004, \quad \frac{m_2}{M} = 1.52880 \pm 0.00004$$
$$\frac{m}{M} = \frac{1}{2}\left(\frac{m_1}{M} + \frac{m_2}{M}\right) = 1.52892 \pm 0.00004$$

As the $m/M \geq 3/2$, we proceed to calculate the critical angle and perform the experiment. Given the ratio, we attempt to calculate the critical angle at which the ring is expected to rise using eqn. (5) in the last section,

$$\cos\theta_c = 0.3792 \pm 0.0004$$
$$\theta_c = 1.1824 \pm 0.0004 \text{ or } 67.76^\circ \pm 0.02^\circ$$

We dropped first and second bead from 70° and 60° and can see a noticeable jump at $\theta_{actual,1} \approx 50^\circ$ and $\theta_{actual,2} \approx 30^\circ$.

We average the initial and final angles of the beads for one averaged bead to make comparison simpler.

$\theta_0 = 65° \pm 3°, \quad \theta_{actual,avg} = 40° \pm 3°$

We used the Angle Measurement website with a protractor to measure the angles in the image taken from our experiment.

We compare the calculated critical angle with the actual critical angle obtained when performing the experiment by finding the difference between initial and final angles in the calculated and actual :

$$\Delta\theta_{actual} = 25° \pm 3°, \quad \Delta\theta_c = 22.24° \pm 0.02°$$

We have the recording of the jump happening in our experiment below,

@ 8_012_proj

We developed a python program with the help of VPython (python library) to create the simulation and represent the expected motion and graph. We have a video representation of the simulation resulting from our code. We used force equation on the rings and bead to calculate the motion at a time increment dt.

@ 8_012_proj_sim.mp4

Discussion

We faced many issues in creating the models and reaching our desired results based on what we had solved theoretically using the measurement we obtained.

- There was a friction force and air resistance which may have accounted for the loss of energy or deviation in our results. This was an issue experimentally as well because we had to find a way to minimize the friction force while not changing the factors that we wanted constantly.
- In our code, we used a time step dt, rather than use a limit at dt approaching zero, we had to assume a fairly small dt (0.001). Going any lower caused issues with the code and going any higher greatly reduced the precision of our results (over stepping). This stems from the fact that it was an Euler integration method, making it a purely computational aspect rather than an issue with the physics we calculated.
- There was a fair amount of difficulty in how we would visualize this code with the graphics, using unfamiliar libraries and having no close blueprint to work with.
- Our original 3D printed prototype contained errors, with the beads not being able to fit into the ring, drilling the beads reduced the mass.
- There was not a precision scale that we could easily find (in two makerspaces), we were forced to initially use very old equipment until we obtained access to a precision scale via a connection

We successfully found that the conditions derived in the original problem were satisfied in both of our models within the error bar (moreso in the computational model). This is significant since verifying satisfies an unintuitive series of steps.

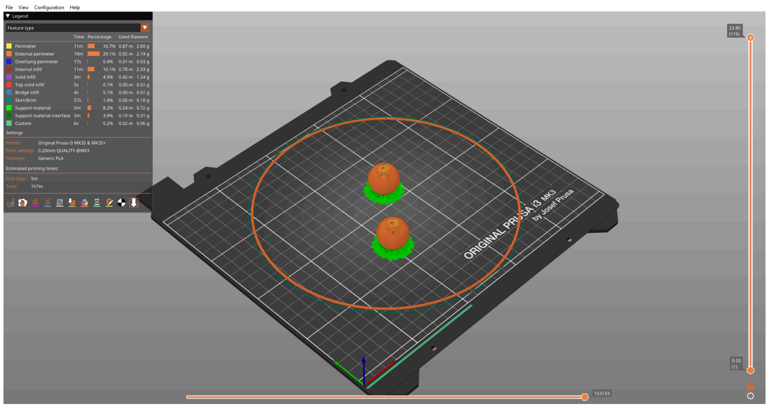

Figure: 3D print model ready in PRUSA software